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On the linearity of on-line computable functions

Hratchia Pélibossian

LITP, Université Paris 7 - Denis Diderot, 2, Place Jussieu, 75251 Paris, Cedex 05, France

Abstract

In this paper, we study the linearity of on-line functions computable by finite automata. Two necessary and sufficient conditions are given for ensuring the linearity of on-line functions computable by finite automata.

We provide an example of a non-affine on-line function, computed by a finite automaton.

0. Introduction

The implementation of algorithms for manipulating the real numbers has always met with difficulties. With the aim of seeking increasingly high-performance circuits, researchers have been led to come up with fundamentally new algorithms.

One of the methods that has been thought of was based on the Avizienis number system [1] where digits may be negative. In this system, it is possible to avoid propagating the carries in an interesting and efficient way. This is the foundation of on-line arithmetic which was introduced by Ercegovac and Trivedi in 1977 [4].

On-line computations are carried out from left to right, digit by digit. The jth digit of the result is computed from the first $\delta + j$ digits of the operands where δ is a constant, called the on-line *delay*.

In [5], Jean-Michel Muller considers functions which are on-line computable by finite automata and he proves the following results.

Theorem 0.1. *Let f be a function of one variable, with a piecewise continuous second derivative. If f is on-line computable by a finite automaton, then in each interval where f'' is continuous, f is an affine function of the form $f(x) = ax + b$, where a, b are rational numbers.*

Theorem 0.2. *Let f be a function of one variable, with a piecewise continuous second derivative. If f is on-line computable by a finite automaton, then the breakpoints of f, i.e. the points where the second derivative of f is not continuous, are rational numbers.*

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This paper deals with linearity conditions for functions which are on-line computable by finite automata.

1. Three results about on-line computable functions

Definition 1. Let

$$D_a = \{-a, -a+1, \ldots, -1, 0, 1, 2, \ldots, a-1, a\},$$

where

$$\frac{r-1}{2} \leqslant a \leqslant r-1.$$

Let

$$\beta = \sum_{i=1}^{\infty} \beta_i r^{-i} \quad \text{where } \beta_i \in D_a.$$

$\sum_{i=1}^{\infty} \beta_i r^{-i}$ is a representation of the number β in the Avizienis system with radix r and digit-set D_a.

In this paper, we consider numbers which are represented in radix $r = 2$ with $a = 1$ and we denote

$$\sum_{i=1}^{\infty} \beta_i r^{-i} \quad \text{by } 0, \beta_1 \beta_2 \ldots \beta_n \ldots$$

Recall that, for any alphabet Σ, Σ^* is the free monoid generated by Σ, and ε denotes the empty word.

We now define sequential machines. A sequential machines is a finite automaton which is endowed with an impression function, i.e. for any state q and input symbol x_0 the machine will perform a transition, and will print output symbols $y_1, y_2, \ldots, y_i$ [4].

Definition 2. A sequential machine is given by

(1) two finite alphabets Σ and Γ;
(2) a finite nonempty set of states Q;
(3) a transition function $\lambda : Q \times \Sigma \to Q$;
(4) an impression function $\sigma : Q \times \Sigma \to \Gamma^*$;
(5) an initial state denoted by q_1.

The transition function λ enables us to define a function $\tilde{\lambda} : Q \times \Sigma^* \to Q$, extending the function λ to words and defined by induction as

(1) $\tilde{\lambda}(q, \varepsilon) = q$;
(2) $\tilde{\lambda}(q, wa) = \lambda(\tilde{\lambda}(q, w), a)$ if $a \in \Sigma$ and $w \in \Sigma^*$.

Similarly, the impression function σ can be extended to a function $\tilde{\sigma} : Q \times \Sigma^* \to \Gamma^*$ $\tilde{\sigma}(q, wa) = \sigma(\tilde{\lambda}(q, w), a)$.

In this paper we suppose that $\Sigma = \Gamma = \{\bar{1}, 0, 1\}$ and $\sigma(q_i, a) \in \{\varepsilon, \bar{1}, 0, 1\}$, where the symbol $\bar{1}$ is another denotation of -1, and ε is the empty word.

For any given sequence of inputs $x_1, x_2, \ldots, x_n, \ldots$, the sequence $y_1, y_2, \ldots, y_n, \ldots$ defined by $y_n = \tilde{\sigma}(q_1, x_1 x_2 \ldots x_n)$, $n \in N$, is called the sequence of outputs computed by the machine A for the sequence of inputs $x_1, x_2, \ldots, x_n, \ldots$

Definition 3. Let f be a function from $[-1,1]$ to $[-1,1]$. We say that the sequential machine A computes the on-line function f with delay δ, where δ is a nonnegative natural number $\delta \geqslant 0$, if for the sequence of inputs $x_1, x_2, \ldots, x_p$, $x_i \in \{-1, 0, 1\}$, machine A computes the sequence of outputs $y_1, y_2, \ldots, y_p$, where $y_i = \varepsilon$ if $i \leqslant \delta$, and $y_i \in \{-1, 0, 1\}$ if $i > \delta$, and if moreover inputs and outputs are such that denoting

$$x = 0, x_1 x_2 \ldots x_n \ldots \quad \text{and} \quad y = 0, y_{\delta+1} y_{\delta+2} \ldots y_n \ldots$$

we have

$$f(x) = y.$$

We consider sequential machines running without delay.

Definition 4. Let A be a sequential machine. We say that the sequential machine A has a state period for input $x = 0, x_1 x_2 \ldots x_m \ldots$ if there is a sequence q_j of states of A, $1 \leqslant j \leqslant k$ such that for the sequence of inputs $x_1, x_2, \ldots, x_m, \ldots$ an integer n can be found, such that for all integers s, l with $l \leqslant k$,

$$q_l = \tilde{\lambda}(q_1, x_1 x_2 \ldots x_n \ldots x_{n+l}) = \tilde{\lambda}(q_1, x_1 x_2 \ldots x_n \ldots x_{n+sk+l})$$

holds.

The number k is said to be the length of this state period, n is said to be the beginning of the state period, and $q_1, q_2, \ldots, q_k$ is said to be the state period for input $x = 0, x_1 x_2 \ldots x_m \ldots$

Let A be a sequential machine with only one input. Let us assume that if we give the number $x = 0, x_1 x_2 \ldots x_n \ldots x_l \ldots$ as input to A, there is a state Φ of machine A which occurs twice: i.e. there are two integers n, l with $n < l$ such that

$$\Phi = \tilde{\lambda}(q_1, x_1 x_2 \ldots x_n) = \tilde{\lambda}(q_1, x_1 x_2 \ldots x_n \ldots x_l). \tag{1}$$

Let $p = l - n$. Then write $t_i = x_{n+i}$, where i is an integer and $i \leqslant p$. Consider the number $\tilde{x} = 0, x_1 \ldots x_n t_1 t_2 \ldots t_p t_1 t_2 \ldots t_p \ldots$. Obviously, $\tilde{x}$ is a periodic number of period $t_1 t_2 \ldots t_p$, length p and with beginning n.

Lemma 1. *Let $x = 0, x_1 x_2 \ldots x_n \ldots x_l \ldots$ be a real number such that*

$$\tilde{\lambda}(q_1, x_1 x_2 \ldots x_n) = \tilde{\lambda}(q_1, x_1 x_2 \ldots x_n \ldots x_l).$$

If the real number

$$\tilde{x} = 0, x_1 \dots x_n t_1 t_2 \dots t_p \dots t_1 t_2 \dots t_p \dots,$$

where $t_i = x_{n+i}$ *for* $1 \leqslant i \leqslant p = l - n$, *is given as input to the sequential machine* A, *then* A *has a state period of length* p *with beginning* n, *and the number* $\beta = 0, \beta_1 \beta_2 \dots \beta_n \dots$ *which is output by* A *is periodic with a period of length* p *and with beginning* n.

The proof of Lemma 1 is easy and is omitted here.

Now, suppose that the sequential machine A computes a given function f. Then, for the number $\tilde{x} = 0, x_1 x_2 \dots x_n t_1 t_2 \dots t_p \dots t_1 t_2 \dots t_p \dots$ and for the function f the following lemma holds.

Lemma 2. *Let* A *be a sequential machine computing function* f, *and let* $x = 0, x_1 x_2 \dots x_n \dots x_l \dots$ *be a real number such that*

$$\tilde{\lambda}(q_1, x_1 x_2 \dots x_n) = \tilde{\lambda}(q_1, x_1 x_2 \dots x_n \dots x_l).$$

If f *has a derivative at*

$$\tilde{x} = 0, x_1 x_2 \dots x_n t_1 t_2 \dots t_p t_1 t_2 \dots t_p \dots,$$

where $t_i = x_{n+i}$ *for* $1 \leqslant i \leqslant p = l - n$, *then* f *is affine on the closed interval*

$$I = [0, x_1 x_2 \dots x_n \bar{1}\bar{1}\bar{1} \dots \ , \ 0, x_1 x_2 \dots x_n 111 \dots].$$

Proof. Notice that if the hypotheses of Lemma 2 are satisfied, then those of Lemma 1 are satisfied too and, consequently, A has a state period of length p with beginning n at number $\tilde{x}$ and it maps number $\tilde{x}$ onto $\beta = 0, \beta_1 \beta_2 \dots \beta_n \alpha_1 \alpha_2 \dots \alpha_p \alpha_1 \alpha_2 \dots \alpha_p \dots$ which is periodic with a period of length p and beginning n.

We thus have $f(\tilde{x}) = \beta$. Consider an arbitrary number y belonging to the interval I. The number y can be written as follows:

$$y = 0, x_1 x_2 \dots x_n y_1 y_2 \dots y_m \dots, \quad \text{where } y_i \in \{\bar{1}, 0, 1\}.$$

Consider now the following sequence of numbers:

$$\begin{aligned}
z_0 &= y = 0, x_1 x_2 \dots x_n y_1 y_2 \dots y_m \dots, \\
z_1 &= 0, x_1 x_2 \dots x_n t_1 t_2 \dots t_p y_1 y_2 \dots y_m \dots, \\
z_2 &= 0, x_1 x_2 \dots x_n t_1 t_2 \dots t_p t_1 t_2 \dots t_p y_1 y_2 \dots y_m \dots, \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
z_i &= 0, x_1 x_2 \dots x_n \underbrace{t_1 t_2 \dots t_p \dots t_1 t_2 \dots t_p}_{i \times p} y_1 y_2 \dots y_m \dots, \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

For each i, consider $\tilde{x} - z_i$ and $f(\tilde{x}) - f(z_i)$:

$$\begin{aligned}\tilde{x} - z_i &= 0,x_1x_2\ldots x_nt_1t_2\ldots t_pt_1t_2\ldots t_p\ldots\\&\quad -0,x_1x_2\ldots x_n\underbrace{t_1t_2\ldots t_p\ldots t_1t_2\ldots t_p}_{i\times p}y_1y_2\ldots y_m\ldots\\&= 0,\underbrace{00\ldots 0}_{n+i\cdot p}t_1t_2\ldots t_pt_1t_2\ldots t_pt_1t_2\ldots t_p\ldots - 0,\underbrace{00\ldots 0}_{n+i\cdot p}y_1y_2\ldots y_m\ldots\\&= 0,\underbrace{00\ldots 01}_{n+i\cdot p}\cdot(0,t_1t_2\ldots t_pt_1t_2\ldots t_p\ldots - 0,y_1y_2\ldots y_m\ldots).\end{aligned}$$

As we have already noticed,

$$f(\tilde{x}) = \beta = 0,\beta_1\beta_2\ldots\beta_n\alpha_1\alpha_2\ldots\alpha_p\alpha_1\alpha_2\ldots\alpha_p\ldots$$

and

$$f(y) = 0,\beta_1\beta_2\ldots\beta_n\gamma_1\gamma_2\ldots\gamma_m\ldots$$

Taking into account that

$$\tilde{\lambda}(q_1,x_1x_2\ldots x_n) = \tilde{\lambda}(q_1,x_1x_2\ldots x_n\underbrace{t_1t_2\ldots t_p\ldots t_1t_2\ldots t_p}_{i\times p}),$$

it ensues that for each integer i, $i \leqslant 0$,

$$f(z_i) = 0,\beta_1\beta_2\ldots\beta_n\underbrace{\alpha_1\alpha_2\ldots\alpha_p\ldots\alpha_1\alpha_2\ldots\alpha_p}_{i\times p}\gamma_1\gamma_2\ldots\gamma_m\ldots$$

Consequently,

$$\begin{aligned}f(\tilde{x}) - f(z_i) &= 0,\beta_1\beta_2\ldots\beta_n\alpha_1\alpha_2\ldots\alpha_p\alpha_1\alpha_2\ldots\alpha_p\ldots\\&\quad -0,\beta_1\beta_2\ldots\beta_n\underbrace{\alpha_1\alpha_2\ldots\alpha_p\ldots\alpha_1\alpha_2\ldots\alpha_p}_{i\times p}\gamma_1\gamma_2\ldots\gamma_m\ldots\\&= 0,\underbrace{00\ldots 0}_{n+i\cdot p}\alpha_1\alpha_2\ldots\alpha_p\alpha_1\alpha_2\ldots\alpha_p\ldots - 0,\underbrace{00\ldots 0}_{n+i\cdot p}\gamma_1\gamma_2\ldots\gamma_m\ldots\\&= 0,\underbrace{00\ldots 01}_{n+i\cdot p}\cdot(0,\alpha_1\alpha_2\ldots\alpha_p\alpha_1\alpha_2\ldots\alpha_p\ldots - 0,\gamma_1\gamma_2\ldots\gamma_m\ldots).\end{aligned}$$

Now notice that

$$\frac{f(\tilde{x}) - f(z_i)}{\tilde{x} - z_i} = \frac{0,\overbrace{00\ldots 01}^{n+i\cdot p}\cdot(0,\alpha_1\alpha_2\ldots\alpha_p\alpha_1\alpha_2\ldots\alpha_p\ldots - 0,\gamma_1\gamma_2\ldots\gamma_m\ldots)}{0,\underbrace{00\ldots 01}_{n+i\cdot p}\cdot(0,t_1t_2\ldots t_pt_1t_2\ldots t_p\ldots - 0,y_1y_2\ldots y_m\ldots)} = K.$$

It can be seen that $(f(\tilde{x}) - f(z_i))/(\tilde{x} - z_i)$ is the same for every integer i. Hence,

$$\lim_{i \to \infty} \frac{f(\tilde{x}) - f(z_i)}{\tilde{x} - z_i} = K,$$

and from the existence of the derivative at $\tilde{x}$, we get

$$f'(\tilde{x}) = K.$$

Besides, setting $i = 0$,

$$\frac{f(\tilde{x}) - f(z_0)}{\tilde{x} - z_0} = \frac{f(\tilde{x}) - f(y)}{\tilde{x} - y} = K$$

is also obtained.

Consequently,

$$f(y) = (y - \tilde{x})K + f(\tilde{x}) = Ky + f(\tilde{x}) - K\tilde{x}.$$

Therefore, $f(y) = Ky + f(\tilde{x}) - K\tilde{x}$, where $\tilde{x}, K, f(\tilde{x})$, are constants, holds for any y in the interval I. Hence, the lemma is proved. □

Definition 3. Let A be a sequential machine and let M be the number of states of machine A. The periodic number

$$\tilde{x} = 0,x_1x_2 \ldots x_n t_1 t_2 \ldots t_p t_1 t_2 \ldots t_p \ldots,$$

where $1 \leqslant n < n + p \leqslant M + 1$, is called test number for A if A has a state period of length p with beginning n for input $\tilde{x}$.

Remark. For any sequential machine A, the number of test numbers for machine A is finite.

The interval I defined as

$$I = [0,x_1x_2 \ldots x_n\overline{1}\overline{1}\overline{1} \ldots,\ \ 0,x_1x_2 \ldots x_n 111 \ldots]$$

is called the interval adapted to $\tilde{x} = 0,x_1x_2 \ldots x_n t_1 t_2 \ldots t_p t_1 t_2 \ldots t_p \ldots$

Theorem 1. *Assume that f is on-line computable by a sequential machine A. If f has first-order derivatives at all test numbers of A, then f is affine in $[-1, 1]$.*

Proof. Let M be the number of states of machine A. We consider a sequence $x_1, x_2, \ldots, x_M, x_{M+1}$, $i \leqslant M + 1$, $x_i \in \{0, 1\}$. Giving this sequence as input to the machine A, we see that there is a state Φ_0 of A which occurs twice, because the number of states of A is equal to M. Let

$$\Phi_0 = \lambda(q_1, x_1x_2 \ldots x_n) = \lambda(q_1, x_1x_2 \ldots x_n x_{n+1} \ldots x_l).$$

Consider the real number

$$\tilde{x} = 0,x_1x_2 \ldots x_n x_{n+1} \ldots x_l x_{n+1} \ldots x_l \ldots$$

It is a test number for A because it is a periodic number with period $x_{n+1}\dots x_l$, length $l-n$, and beginning n, where $n < l \leqslant M+1$. The number $\tilde{x}$, the machine A and the function f satisfy the conditions of Lemma 2. Therefore, f is affine on the interval

$$I = [0,x_1x_2\dots x_n\overline{111}\dots,\ 0,x_1x_2\dots x_n111\dots]$$

Now consider all finite sequences $x_1,x_2,\dots,x_M,x_{M+1}$, where $x_i \in \{0,1\}$, for $i \leqslant M+1$. There are 2^{M+1} different sequences $x_1,x_2,\dots,x_M,x_{M+1}$, namely,

$$\begin{array}{l}
x_{1,1},x_{1,2},\dots,x_{1,M+1},\\
x_{2,1},x_{2,2},\dots,x_{2,M+1},\\
\quad\vdots\quad\ \vdots\qquad\vdots\\
x_{i,1},x_{i,2},\dots,x_{i,M+1},\\
\quad\vdots\quad\ \vdots\qquad\vdots\\
x_{2^{M+1},1},x_{2^{M+1},2},\dots,x_{2^{M+1},M+1}.
\end{array}$$

Let

$$x^{(i)} = 0,x_{1,1}x_{1,2}\dots x_{1,M+1}00\dots0\dots,\quad \text{where } 1 \leqslant i \leqslant 2^{M+1}.$$

It is assumed that if $i \leqslant j$, then $x^{(i)} \leqslant x^{(j)}$.

Notice that

$$x^{(i+1)} - x^{(i)} = \frac{1}{2^{M+1}},\quad \text{where } 1 \leqslant i \leqslant 2^{M+1}.$$

For each sequence $x_{i,1},x_{i,2},\dots,x_{i,M+1}$ define the real number

$$y^{(i)} = 0,x_{i,1}x_{i,2}\dots x_{i,n_i}x_{i,n_i+1}\dots x_{i,l_i}x_{i,n_i+1}\dots x_{i,l_i}\dots,$$

where

$$1 \leqslant n_i < l_i \leqslant M+1.$$

With $y^{(i)}$ as input, A has a state period of length $l_i - n_i$ and with beginning n_i. Obviously, all the $y^{(i)}, 1 \leqslant i \leqslant 2^{M+1}$, are test numbers. Hence, for every $y^{(i)}, 1 \leqslant i \leqslant 2^{M+1}$, $f'(y^{(i)})$ is defined and consequently f is affine on

$$R_i = [0,x_{i,1}x_{i,2}\dots x_{i,n_i}\overline{111}\dots\ ,\ 0,x_{i,1}x_{i,2}\dots x_{i,n_i}111\dots],\quad 1 \leqslant i \leqslant 2^{M+1}.$$

That is to say if $z \in R_i$,

$$f(z) = K_iz + B_i,$$

where K_i, B_i are constants.

For every i, $i \leqslant 2^{M+1}-1$, consider the intervals

$$I_i = [0,x_{i,1}x_{i,2}\dots x_{i,M+1}\overline{111}\dots,\ 0,x_{i,1}x_{i,2}\dots x_{i,M+1}111\dots]$$

and

$$I_{i+1} = [0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}\overline{111}\dots,\ \ 0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}111\dots].$$

Obviously,

$$I_i \subseteq R_i \quad \text{and} \quad I_{i+1} \subseteq R_{i+1} \quad \text{since} \quad n_i \leqslant M+1 \quad \text{and} \quad n_{i+1} \leqslant M+1 \quad .$$

Consider the real numbers

$$x^{(i)} = 0, x_{i,1}x_{i,2}\dots x_{i,M+1}000\dots \quad \text{and} \quad x^{(i+1)} = 0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}000\dots$$

Notice that

$$x^{(i)} < x^{(i+1)} \quad \text{and that} \quad x^{(i+1)} - x^{(i)} = \frac{1}{2^{M+1}}.$$

Hence,

$$x^{(i+1)} = x^{(i)} + \frac{1}{2^{M+1}}.$$

It follows that

$$\begin{aligned} 0, x_{i,1}x_{i,2}\dots x_{i,M+1}111\dots &= 0, x_{i,1}x_{i,2}\dots x_{i,M+1}000\dots + 0,\underbrace{00\dots0}_{M+1}111\dots \\ &= x^{(i)} + \frac{1}{2^{M+1}} = x^{(i+1)}, \end{aligned}$$

consequently,

$$\begin{aligned} I_i \cap I_{i+1} &= [0, x_{i,1}x_{i,2}\dots x_{i,M+1}\overline{111}\dots, 0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}000\dots] \\ &\quad \cap [0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}\overline{111}\dots, 0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}111\dots] \\ &= [0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}\overline{111}\dots, 0, x_{i+1,1}x_{i+1,2}\dots x_{i+1,M+1}000\dots]. \end{aligned}$$

Therefore $I_i \cap I_{i+1}$ is a nonempty interval which is not reduced to a single point. But

$$I_i \cap I_{i+1} \subseteq R_i \cap R_{i+1}$$

since

$$I_i \subseteq R_i \text{ and } I_{i+1} \subseteq R_{i+1}.$$

Therefore, $R_i \cap R_{i+1}$ also is a nonempty interval which is not reduced to a single point. Function f is affine on the intervals R_i and R_{i+1} and so

$$f(y) = \begin{cases} K_i y + B_i & \text{if } y \in R_i, \\ K_{i+1} y + B_{i+1} & \text{if } y \in R_{i+1}. \end{cases}$$

There are two real numbers v and w such that

$$v, w \in R_i \cap R_{i+1} \quad \text{and} \quad v \neq w.$$

Thus,

$$f(v) = K_i v + B_i, \quad f(v) = K_{i+1} v + B_{i+1},$$
$$f(w) = K_i w + B_i, \quad f(w) = K_{i+1} w + B_{i+1}.$$

Therefore,

$$K_i = K_{i+1} \quad \text{and} \quad B_i = B_{i+1}.$$

Equalities

$$K_1 = K_2 = \cdots = K_{2^{M+1}} \quad \text{and} \quad B_1 = B_2 = \cdots = B_{2^{M+1}}$$

are obtained, and for $u \in \bigcup_{i=1}^{2^{M+1}} R_i$,

$$f(u) = K_1 u + B_1.$$

As $I_i \subseteq R_i$, $\bigcup_{i=1}^{2^{M+1}} I_i \subseteq \bigcup_{i=1}^{2^{M+1}} R_i$ holds. Hence, for every $v \in \bigcup_{i=1}^{2^{M+1}} I_i$

$$f(v) = K_1 \cdot v + B_1.$$

We thus obtain that f is affine on

$$\bigcup_{i=1}^{2^{M+1}} I_i = [0, \underbrace{00\ldots0}_{M+1}\overline{111}\ldots \quad , \quad 1],$$

and in the same way, f is affine on

$$[-1, 0, \underbrace{00\ldots0}_{M+1} 111\ldots].$$

Finally, we obtain that f is affine on $[-1, 1]$. Theorem 1 is thus proved. □

Theorem 2. *Assume that function f is on-line computable by a sequential machine A. If for all the test numbers x_i, the points $(x_i, f(x_i))$ are on the same line then f is affine in $[-1, 1]$.*

Proof. Let D be the set of numbers such that for every number $x \in D$, $(x, f(x))$ is on the same line as the points $(x_i, f(x_i))$ for x_i ranging over the set of test numbers. Let n be fixed and let $x = 0, x_{0,1}x_{0,2}\ldots x_{0,n}x_{0,n+1}\ldots$ be a real number such that $x_{0,n} \neq 0$ and for every integer j with $j > n$ $x_{0,j} = 0$, thus

$$x = 0, x_{0,1}x_{0,2}\ldots x_{0,n}000\ldots$$

If $x \in D$ is proved, then the set of numbers D will be proved to be everywhere dense and, from the continuity of on-line computable functions [2], it will follow that f is affine on the interval $[-1, 1]$.

Now, construct, by induction, two finite sequences.

Step 0: Let

$$z_0 = x = 0, x_{0,1}x_{0,2}\ldots x_{0,n}\ldots$$

Consider the sequence $x_{0,1}, x_{0,2}, \ldots, x_{0,M+1}$. Because the number of states of A is equal to M, if the sequence $x_{0,1}, x_{0,2}, \ldots, x_{0,M+1}$ is given as input to A, one can see that there is a state Φ_0 of machine A which must occur twice.

Let

$$\Phi_0 = \lambda(q_1, x_{0,1}x_{0,2}\ldots x_{0,s}) = \lambda(q_1, x_{0,1}x_{0,2}\ldots x_{0,s}x_{0,s+1}\ldots x_{0,r}),$$

where $s < r \leqslant M+1$. Define

$$y_0 = 0, x_{0,1}x_{0,2}\ldots x_{0,s}x_{0,s+1}\ldots x_{0,r}x_{0,s+1}\ldots x_{0,r}\ldots$$

Obviously, y_0 is a test number. If z_0 is a test number, the construction of the sequences $\{z_j\}_{j=0}^m$ and $\{y_j\}_{j=0}^m$ is completed. Otherwise construct z_1 from z_0 by removing the sequence of digits $x_{0,s+1}x_{0,s+2}\ldots x_{0,r}$.

From this we obtain

$$z_1 = 0, x_{0,1}x_{0,2}\ldots x_{0,s}x_{0,r+1}x_{0,r+2}\ldots,$$

and denote

$$z_1 = 0, x_{1,1}x_{1,2}\ldots x_1, sx_1, s+1\ldots$$

Step p:

$$z_p = 0, x_{p,1}x_{p,2}\ldots x_{p,n}\ldots$$

Consider the sequence $x_{p,1}, x_{p,2}, \ldots, x_{p,M+1}$. Giving this sequence as input to machine A, it is likewise seen that, since the number of states of machine A is equal to M, there is a state Φ_p of machine A which must occur twice.

Let

$$\Phi_p = \lambda(q_1, x_{p,1}x_{p,2}\ldots x_{p,k}) = \lambda(q_1, x_{p,1}x_{p,2}\ldots x_{p,k}x_{p,k+1}\ldots x_{p,l}).$$

Let us take

$$y_p = 0, x_{p,1}x_{p,2}\ldots x_{p,k}x_{p,k+1}\ldots x_{p,l}x_{p,k+1}\ldots x_{p,l}\ldots,$$

where $k < l \leqslant M+1$. Obviously, y_p is a test number. If z_p is a test number, then the construction is completed. Otherwise we construct z_{p+1} from z_p by removing the sequence of digits $x_{p,k+1}x_{p,k+2}\ldots x_{p,l}$, and we obtain

$$z_{p+1} = 0, x_{p,1}x_{p,2}\ldots x_{p,k}x_{p,l+1}x_{p,l+2}\ldots$$

Denote

$$z_{p+1} = 0, x_{p+1,1}x_{p+1,2}\ldots x_{p+1,k}x_{p+1,k+1}\ldots$$

Now, let us prove that both sequences $\{z_j\}_{j=0}^m$ and $\{y_j\}_{j=0}^m$ are finite. Assume this is not the case. If the sequences are infinite, then for every j, z_j is not a test number. We also have that

$$x = 0, x_{0,1}x_{0,2}\ldots x_{0,n}000\ldots,$$

and at every step j we have

$$z_j = 0, x_{j,1}x_{j,2}\dots x_{j,s_j}000\dots$$

Obviously, at least one of the $x_{0,i}$, $i=1,2,\dots,n$, is removed at every step j, $j=1,2,\dots$ Otherwise, if it is assumed that at step $p+1$ no $x_{0,i}$ is removed $i=1,2,\dots,n$, the following would hold:

$$z_p = 0, x_{p,1}x_{p,2}\dots x_{p,n}\dots,$$
$$y_p = 0, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,k+1}\dots x_{p,l}x_{p,k+1}\dots x_{p,l}\dots,$$
$$z_{p+1} = 0, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,l+1}x_{p,l+2}\dots,$$

where $x_{p,r} = x_{0,t}$ with $r > k, t > n$. Consequently, $x_{p,r} = x_{0,t} = 0$, and then

$$z_p = z_{p+1} = y_p = 0, x_{p,1}x_{p,2}\dots x_{p,k}000\dots,$$

and it would follow from this that z_p would be a test number for machine A, and so a contradiction would be reached. Thus at each step, at least one of the $x_{0,i}$, $i=1,2,\dots,n$, is removed.

Since the number of numbers $x_{0,i}$, $i=1,2,\dots,n$, is equal to n, the construction of both sequences $\{z_j\}_{j=0}^{m}$ and $\{y_j\}_{j=0}^{m}$ stops after a finite number of steps $m+1 \leqslant n+1$. Consequently, $\{z_j\}_{j=0}^{m}$ and $\{y_j\}_{j=0}^{m}$ are finite sequences, and z_{m+1} is a test number for A. It is obvious that $z_{p+1} \neq y_p$, $z_{p+1} \neq z_p$, $y_p \neq z_p$ for each p, $p \leqslant m$.

Let us now prove that, for each $p \leqslant m$, the points $(z_p, f(z_p))$, $(y_p, f(y_p))$ and $(z_{p+1}, f(z_{p+1}))$ are on the same line. We have

$$I = [0, x_1x_2\dots x_n\overline{111}\dots, 0, x_1x_2\dots x_n111\dots],$$
$$z_p = 0, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,k+1}\dots x_{p,l}x_{p,l+1}\dots x_{p,n}\dots,$$
$$y_p = 0, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,k+1}\dots x_{p,l}x_{p,k+1}\dots x_{p,l}\dots,$$
$$z_{p+1} = 0, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,l+1}x_{p,l+2}\dots$$

Taking into account that

$$\lambda(q_1, x_{p,1}x_{p,2}\dots x_{p,k}) = \lambda(q_1, x_{p,1}x_{p,2}\dots x_{p,k}x_{p,k+1}\dots x_{p,l}),$$

we also have

$$f(z_p) = 0, \beta_1\beta_2\dots\beta_k\alpha_1\alpha_2\dots\alpha_{l-k}\gamma_1\gamma_2\gamma_3\dots,$$
$$f(y_p) = 0, \beta_1\beta_2\dots\beta_k\alpha_1\alpha_2\dots\alpha_{l-k}\alpha_1\alpha_2\dots\alpha_{l-k}\dots,$$
$$f(z_{p+1}) = 0, \beta_1\beta_2\dots\beta_k\gamma_1\gamma_2\gamma_3\dots$$

From the last three equalities, it follows that

$$\frac{f(y_p) - f(z_p)}{y_p - z_p} = \frac{f(y_p) - f(z_{p+1})}{y_p - z_{p+1}}.$$

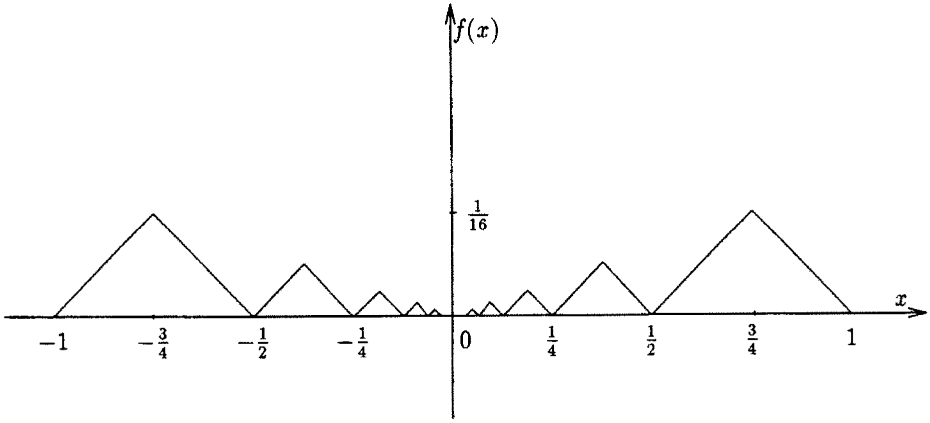


Fig. 1.

Thus, the points $(z_p, f(z_p))$, $(y_p, f(y_p))$ and $(z_{p+1}, f(z_{p+1}))$ are on the same line for $p=1,2,\ldots,m$. Letting $p=m$, it follows that $(z_m, f(z_m))$, $(y_m, f(y_m))$ and $(z_{m+1}, f(z_{m+1}))$ are on the same line; since $z_{m+1} \in D$ and $y_m \in D$, we deduce that $z_m \in D$.

The same conclusion holds for every i, $i \leqslant m$, $z_i \in D$, and thus $z_0 = x \in D$. Theorem 2 is thus proved. □

Theorem 3. *There is a function which is on-line computable by a sequential machine, and which is not a piecewise affine function on* $[-1,1]$.

Consider the function f, defined by (see Fig. 1)

$$f(x) = \begin{cases} 2^{-2}(x + 2^{-i}) & \text{if } x \in [-2^{-i}, -3 \cdot 2^{-(i+2)}], \\ 2^{-2}(-x - 2^{-(i+1)}) & \text{if } x \in [-3 \cdot 2^{-(i+2)}, -2^{-(i+1)}], \\ 2^{-2}(x - 2^{-(i+1)}) & \text{if } x \in [2^{-(i+1)}, 3 \cdot 2^{-(i+2)}], \\ 2^{-2}(2^{-i} - x) & \text{if } x \in [3 \cdot 2^{-(i+2)}, 2^{-i}], \\ 0 & \text{if } x = 0. \end{cases}$$

This function is continuous. Nevertheless, it is not piecewise affine on $[-1,1]$ because there is no interval containing 0, on which the function is affine.

Remark that if

$$x \in \left[-\frac{1}{2^i}, -\frac{1}{2^{i+1}} \right] \cup \left[\frac{1}{2^{i+1}}, \frac{1}{2^i} \right], \quad i = 0, 1, 2, \ldots,$$

then

$$f(x) = 2^{-(i+2)} g_1(y) = 2^{-(i+2)} \cdot 2^{-2}(1 - |y|) \quad \text{where} \quad y = 2^{i+2} \cdot |x| - 3$$

and if

$$x \in \left[-\frac{3}{2^{i+1}}, -\frac{3}{2^{i+2}} \right] \cup \left[\frac{3}{2^{i+2}}, \frac{3}{2^{i+1}} \right], \quad i = 1, 2, \ldots,$$

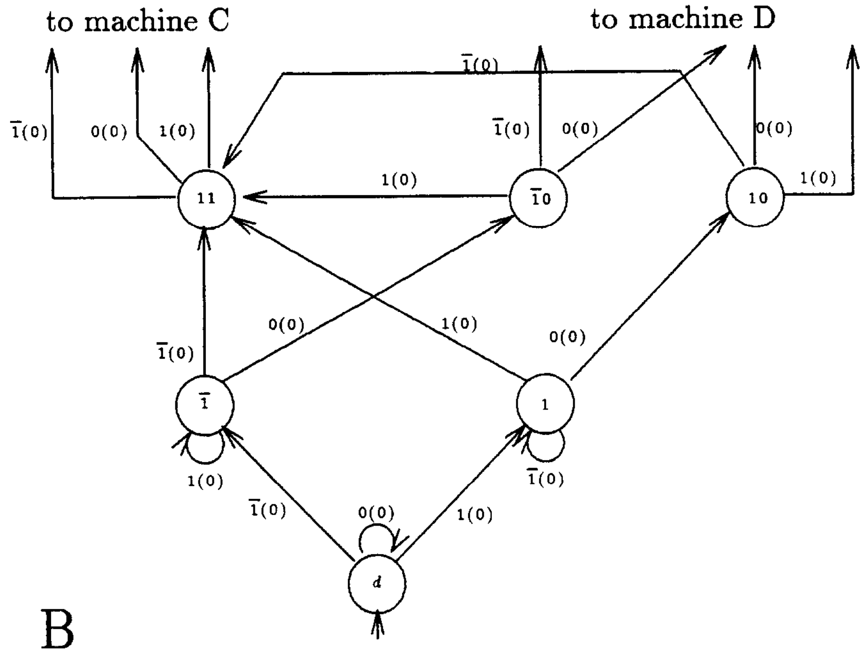


Fig. 2.

then

$$f(x) = 2^{-i} g_2(y) = 2^{-i} 2^{-2} |y| \quad \text{where } y = 2^i \cdot |x| - 1.$$

These alternative definitions of f are used for the construction of the sequential machine A that computes f.

The machine A consists of three submachines, B, C and D. B computes either the coefficient $2^{-(i+2)}$ or the coefficient 2^{-i} and chooses between submachine C and submachine D accordingly.

If B determines that the input x belongs to

$$\left[-\frac{1}{2^i}, -\frac{1}{2^{i+1}} \right] \cup \left[\frac{1}{2^{i+1}}, \frac{1}{2^i} \right]$$

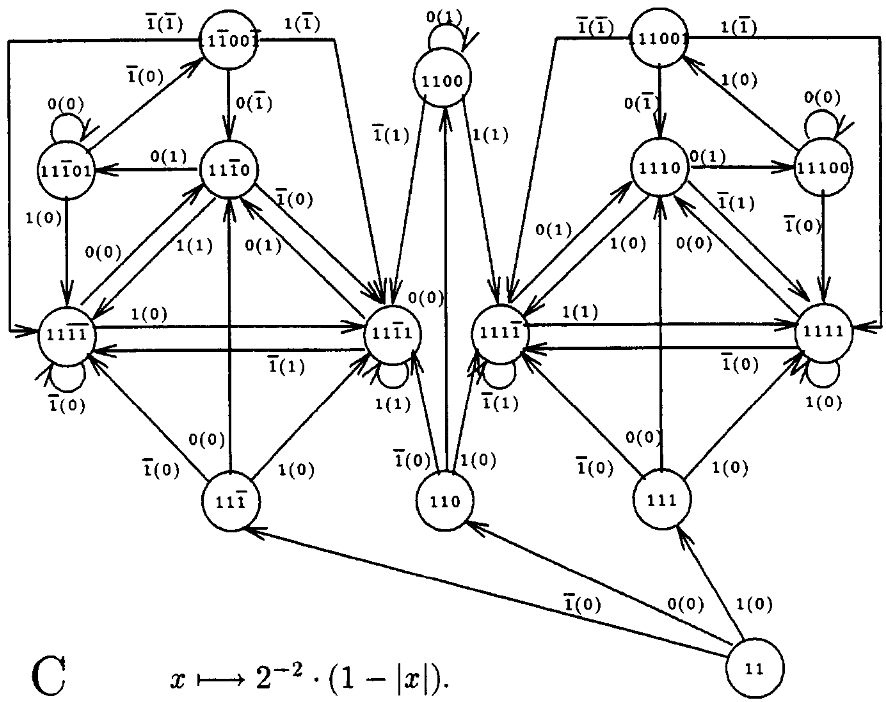
it computes the coefficient $2^{-(i+2)}$, takes state 11, and switches on submachine C.

If B determines that the input x belongs to

$$\left[-\frac{3}{2^{i+1}}, -\frac{3}{2^{i+2}} \right] \cup \left[\frac{3}{2^{i+2}}, \frac{3}{2^{i+1}} \right]$$

it computes the coefficient 2^{-i}, takes state $\bar{1}0$ or 10, and switches on submachine D (see Fig. 2).

Submachine C (Fig. 3) is a sequential machine which computes function $g_1(x) = 2^{-2}(1 - |x|)$.



C $\quad x \longmapsto 2^{-2}\cdot(1-|x|).$

Fig. 3.

Submachine D (Fig. 4) is a sequential machine which computes function $2^{-2}|x|$. Finally, mahcine A computes the function f. The diagram of machine A is given in the appendix.

Example. Let $x=0,11000\ldots$, and $y=0,10111\ldots$ Obviously $x=y$, and $f(x)=f(y)=2^{-4}$.

For input x we have

Step	Input	Output	State
1	1	0	1
2	1	0	11
3	0	0	110
4	0	0	1100
5	0	1	1100
6	0	1	1100
⋮	⋮	⋮	⋮
n	0	1	1100
⋮	⋮	⋮	⋮

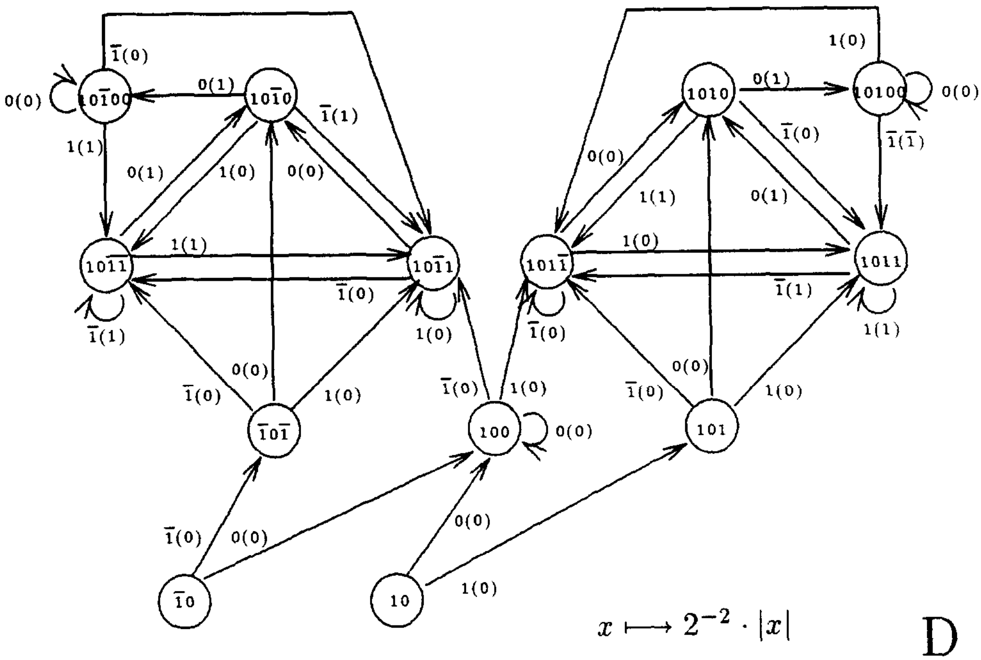


Fig. 4.

The output number is $0,0000111\ldots$, which is equal to $f(x) = 2^{-4}$. Notice that in step 2, the machine determines that $x \in [1, \frac{1}{2}]$.

For input y we have

Step	Input	Output	State
1	1	0	1
2	0	0	10
3	1	0	101
4	1	0	1011
5	1	1	1011
6	1	1	1011
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	1	1011
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The output number is $0,0000111\ldots = 2^{-4}$ and it is equal to the output number computed on input x. The machine determines that $y \in [\frac{3}{4}, \frac{3}{8}]$ at step 2.

2. Conclusion

In this paper we have studied conditions of linearity for functions which are on-line computable by sequential machines. We defined the notion of test numbers for sequential machines which allowed us to find two necessary and sufficient conditions of linearity for such on-line computable functions. These conditions are given by Theorems 1 and 2. Theorem 2 allows to check effectively whether such a given on-line computable function is affine, because, according to the theorem, we only have to check if the values of the function are on the same line as a finite set of points corresponding to rational numbers and this is a decidable problem.

Moreover, Theorem 3 proves that sequential machines are able to compute continuous functions which are not piecewise affine.

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